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k-super cube root cube mean labeling of some corona graphs

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Abstract

Let G be a graph with $|V(G)| = p$ *and* $|E(G)| = q$ *and* $f : V(G) \to \{\mathrm{k},\ \mathrm{k}+1,\ \mathrm{k}+2, \ldots,\ \mathrm{p}+\mathrm{q}+\mathrm{k}-1\ \}$ *be an one-to-one function. The induced edge labeling* f^**, for a vertex labeling f is defined by*

$$f^*(e) = \left\lfloor \sqrt[3]{\frac{f(u)^3 + f(v)^3}{2}} \right\rfloor \quad or \quad \left\lceil \sqrt[3]{\frac{f(u)^3 + f(v)^3}{2}} \right\rceil$$

for all $e = uv \in E(G)$ *is bijective. If* $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{\mathrm{k},\ \mathrm{k}+1,\ \mathrm{k}+2, \ldots,\ \mathrm{p}+\mathrm{q}+\mathrm{k}-1\}$*, then f is known as a k-super cube root cube mean labeling. If such labeling exists, then G is a k-super cube root cube mean graph. In this paper, I prove that* $T_n \odot K_1$, $A(T_n) \odot K_1$, $A(T_n) \odot 2K_1$, $A(Q_n) \odot K_1$, $P_n \odot K_{1,2}$ *and* $P_n \odot K_{1,3}$ *are k-super cube root cube mean graphs.*

Keywords: *K-Super cube root cube mean labeling, Alternate snake graph,* $A(T_n) \odot K_1$, $A(T_n) \odot 2K_1$, $T_n \odot K_1$, $A(Q_n) \odot K_1$, $P_n \odot K_{1,3}$.

MSC (2020): *05C78.*

1. Introduction

In this paper, all graphs are simple, finite, and undirected. Labeling of a graph is an assignment of integers to the vertices or edges or both subject to certain conditions. Several types of graph labeling and an extensive survey are available in [1]. Standard notations of F.Harary [2] are followed here. Somasundaram and Ponraj [5] introduced mean labeling. Let G be a graph with V(G) = p and E(G) = q. A mean labeling f is an injection from V to the set {0, 1, 2, ..., q } such that every edge uv, is labelled with $\frac{[f(u)+f(v)]}{2}$ if [f(u) + f(v)] is even and $\frac{[f(u)+f(v)]+1}{2}$ if [f(u) + f(v)] is odd then the resulting edges are distinct. A graph that accepts a mean labeling is known as mean graph. It was extended to root square mean labeling [7], cube root cube mean labeling [3], etc. Radhika and Vijayan [6] defined a new labeling namely super cube root cube mean labeling. Let f : V(G) → { 1, 2, 3, ... , p + q } be an one to one function. For a vertex labeling f, the induced edge labeling f*, is defined by $f^*(e) = \left\lfloor \sqrt[3]{\frac{f(u)^3+f(v)^3}{2}} \right\rfloor$ or $\left\lceil \sqrt[3]{\frac{f(u)^3+f(v)^3}{2}} \right\rceil$. Then f is known as a super cube root cube mean labeling if f(V(G)) ∪ {f*(e) : e ∈ E(G)} = {1, 2, 3, ..., p + q}. If such labeling exists, then G is a super cube root cube mean graph. Motivated by the concept of super cube root cube mean labeling, Princy Kala [4] introduced a new labeling called k-super cube root cube mean labeling. In this paper, I prove that $T_n \odot K_1$, $A(T_n) \odot K_1$, $A(T_n) \odot 2K_1$, $A(Q_n) \odot K_1$, $P_n \odot K_{1,2}$ and $P_n \odot K_{1,3}$ are k-super cube root cube mean graph. Consider a graph G with $p = |V(G)|$ and $q = E(G)$ and $f : V(G) \to \{k, k+1, k+2, \ldots, p+q+k-1\}$ be an one-to-one function. For a vertex labeling f the induced edge labeling f*, is defined by $f^*(e) = \left\lfloor \sqrt[3]{\frac{f(u)^3+f(v)^3}{2}} \right\rfloor$ or $\left\lceil \sqrt[3]{\frac{f(u)^3+f(v)^3}{2}} \right\rceil$ for all $e = uv \in E(G)$ is bijective. If $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{k, k+1, k+2, \ldots, p+q+k-1\}$, then f is said to be a k-super cube root cube mean labeling. If such a labeling exists, then G is a k-super cube root cube mean graph. Throughout this paper, assumed that k is an integer and ≥ 1.

2. Preliminaries

Definition 2.1. *The triangular snake graph is obtained from a path* $u_1, u_2, \ldots, u_n$ *by joining* u_i *and* u_{i+1} *to a new vertex* v_i, $1 \le i \le n-1$.

Definition 2.2. *To construct an alternate triangular snake graph $A(T_n)$, we have to join u_i and u_{i+1} (alternately) from a path with vertices u_1, u_2, $\dots$, u_n to a vertex v_j, for $1 \leq i \leq n-1$ & $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$.*

Definition 2.3. *An alternate quadrilateral snake graph $A(Q_n)$ is obtained from a path $v_1, v_2, \dots, v_n$ by joining v_i and v_{i+1} (alternately) to the vertices x_j, y_j respectively then joining x_j and y_j , for $1 \leq i \leq n-1$ & $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$.*

Definition 2.4. *The corona of two graphs G and H is formed by taking one copy of G and $|V(G)|$ copies of H, where the j^{th} vertex of G is adjacent to every vertex in the j^{th} copy of H.*

3. Main Results

Theorem 3.1. *The graph $T_n \odot K_1$ is a k-super cube root cube mean graph.*

Proof. Let $G = T_n \odot K_1$
Let $V(G) = \{u_i,\ u_i',: 1 \leq i \leq n\} \cup \{v_i,\ v_i', : 1 \leq i \leq n-1\}$ and
$E(G) = \{v_i\ v_i',\ u_iv_i,\ u_iu_{i+1},\ u_{i+1}v_i : 1 \leq i \leq n-1\} \cup \{u_iu_i',\ 1 \leq i \leq n\}$.
Here $p = |V(G)| = 4n-2$ and $q = |E(G)| = 5n-4$
Hence $p+q = 9n-6$.
Now define a function
$f : V(G) \rightarrow \{k,\ k+1,\ k+2, \dots,\ p+q+k-1\}$ by
$f(u_i) = k + 9i - 7,\ 1 \leq i \leq n$ and $i \neq 2$

$$f(u_2) = \begin{cases} k+9, & k = 1,2,3,4,5; \\ k+11, & otherwise. \end{cases}$$

$f(v_i) = k + 9i - 5,\ 1 \leq i \leq n-1$
$f(u_i') = k + 9i - 9, 1 \leq i \leq n$ and $i \neq 2$

$$f(u_2') = \begin{cases} k+11, & k = 1,2,3,4,5; \\ k+9, & otherwise. \end{cases}$$

$f(v_i') = k + 9i - 3,\ 1 \leq i \leq n-1$
Then the edge labels are
$f^*(u_iu_{i+1}) = k + 9i - 2,\ 1 \leq i \leq n-1$
$f^*(u_iu_i') = k + 9i - 8,\ 1 \leq i \leq n$
$f^*(u_iv_i) = k + 9i - 6,\ 1 \leq i \leq n-1$
$f^*(u_{i+1}v_i) = k + 9i - 1,\ 1 \leq i \leq n-1$
$f^*(v_iv_i') = k + 9i - 4,\ 1 \leq i \leq n-1$
Clearly $f[V(G)] \cup \{f^*(e) : e \in E(G)\} = \{k,\ k+1,\ k+2, \dots,\ p+q+k-1\}$.

Hence $T_n \odot K_1$ is a k-super cube root cube mean graph.
An example of 5-super cube root cube mean labeling of $T_4 \odot K_1$ is shown in Figure 1.

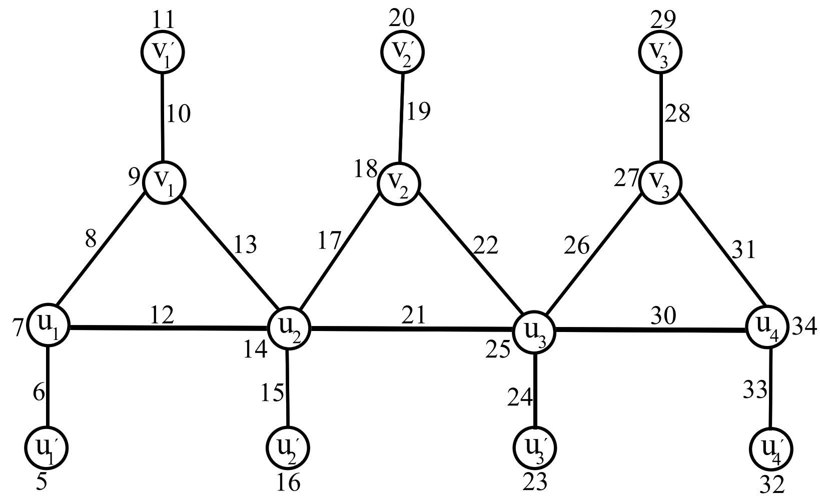

Figure 1: 5-super cube root cube mean labeling of $T_4 \odot K_1$

□

Theorem 3.2. *The graph $A(T_n) \odot K_1$ is a k-super cube root cube mean graph.*

Proof. Let $G = A(T_n) \odot K_1$
Here consider two cases.

Case 1: The triangle starts from u_1.
Let $V(G) = \{ u_i,\ u_i' : 1 \leq i \leq n \} \cup \{ v_i,\ v_i' : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \}$ and
$E(G) = \{ u_i u_{i+1} : 1 \leq i \leq n\text{-}1\} \cup \{ u_{2i-1}v_i,\ u_{2i}v_i,\ v_iv_i' : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \} \cup \{ u_iu_i' : 1 \leq i \leq n \}$.

here, $p = |V(G)| = \begin{cases} 3n, & for\ n\ is\ even \\ 3n-1, & for\ n\ is\ odd. \end{cases}$ and

$q = |E(G)| = \begin{cases} \frac{7n-2}{2}, & for\ n\ is\ even \\ \frac{7n-5}{2}, & for\ n\ is\ odd. \end{cases}$

Therefore p + q = $\begin{cases} \frac{13n-2}{2}, & for\ n\ is\ even \\ \frac{13n-7}{2}, & for\ n\ is\ odd. \end{cases}$

Now define a function

f : V(G) $\rightarrow$ {k, k + 1, k + 2, . . . , p + q + k − 1 } by

$f(u_{2i-1})$ = k + 13i - 11, $1 \leq i \leq \lceil \frac{n}{2} \rceil$.

$f(u_2) = \begin{cases} k+7, & k = 1, 2; \\ k+8, & otherwise. \end{cases}$

$f(u_{2i})$ = k + 13i - 5, $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f(v_i)$ = k + 13i - 9, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f(v'_i)$ = k + 13i - 4, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f(u'_{2i-1})$ = k + 13i - 13, $1 \leq i \leq \lceil \frac{n}{2} \rceil$.

$f(u'_{2i})$ = k + 13i - 2, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Then the edge labels are

$f^*(u_{2i-1}u_{2i})$ = k + 13i -8, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f^*(u_{2i}u_{2i+1})$ = k + 13i -1, $1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.

$f^*(u_{2i-1}v_i)$ = k + 13i - 10, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f^*(u_{2i}v_i)$ = k + 13i - 7, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f^*(v_1v'_1) = \begin{cases} k+8, & k = 1, 2; \\ k+7, & otherwise. \end{cases}$

$f^*(v_iv'_i)$ = k + 13i - 6, $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f^*(u_{2i-1}u'_{2i-1})$ = k + 13i - 12, $1 \leq i \leq \lceil \frac{n}{2} \rceil$.

$f^*(u_{2i}u'_{2i})$ = k + 13i - 3, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Clearly f[V(G)] $\cup$ {f*(e) : e $\in$ E(G)} = {k, k + 1, k + 2, . . . , p + q + k − 1}.

An example of 2-super cube root cube mean labeling of A(T_6) $\odot$ K_1[Triangle starts from u_1] is shown in Figure 2.

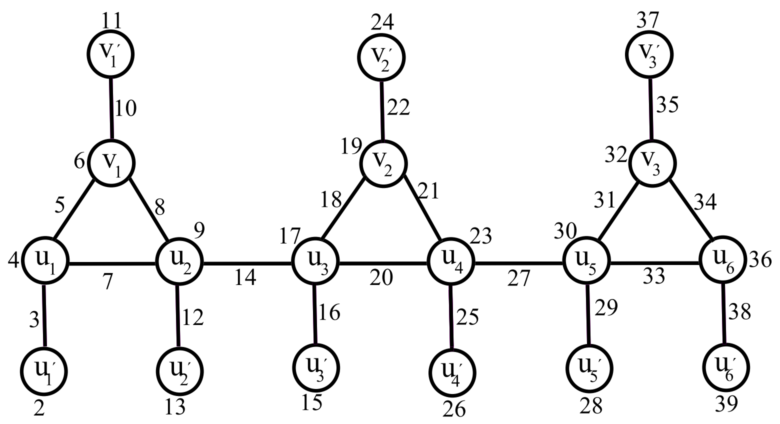

Figure 2: 2-super cube root cube mean labeling of $A(T_6) \odot K_1$ [Triangle starts from u_1]

Case 2: The triangle starts from u_2.
Let $V(G) = \{ u_i,\ u_i' : 1 \leq i \leq n \} \cup \{ v_i,\ v_i' : 1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil \}$ and
$E(G) = \{ u_i u_{i+1} : 1 \leq i \leq n\text{-}1\} \cup \{ u_{2i} v_i,\ u_{2i+1} v_i,\ v_i v_i' : 1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil \} \cup \{ u_i u_i' : 1 \leq i \leq n \}$.

here $p = |V(G)| = \begin{cases} 3n-2, & for\ n\ is\ even \\ 3n-1, & for\ n\ is\ odd. \end{cases}$ and
$q = |E(G)| = \begin{cases} \frac{7n-8}{2}, & for\ n\ is\ even \\ \frac{7n-5}{2}, & for\ n\ is\ odd. \end{cases}$
Therefore $p + q = \begin{cases} \frac{13n-12}{2}, & for\ n\ is\ even \\ \frac{13n-7}{2}, & for\ n\ is\ odd. \end{cases}$

Now define a function $f : V(G) \rightarrow \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$ by
$f(u_1) = k$, for all k
$f(u_{2i-1}) = k + 13i - 14,\ 2 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f(u_{2i}) = k + 13i - 7,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f(v_i)$ = k + 13i - 5, $1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f(v'_i)$ = k + 13i, $1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f(u'_{2i-1})$ = k + 13i - 11, $1 \leq i \leq \lceil \frac{n}{2} \rceil$.

$$f(u'_2) = \begin{cases} k+3, & k = 1, 2, 3, 4; \\ k+4, & otherwise. \end{cases}$$

$f(u'_{2i})$ = k + 13i - 9, $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Then the edge labels are

$$f^*(u_1u_2) = \begin{cases} k+4, & k = 1,2,3,4; \\ k+3, & otherwise. \end{cases}$$

$f^*(u_{2i-1}u_{2i})$ = k + 13i -10, $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(u_{2i}u_{2i+1})$ = k + 13i - 4, $1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(u_{2i}v_i)$ = k + 13i - 6, $1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(u_{2i+1}v_i)$ = k + 13i - 3, $1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(v_iv'_i)$ = k + 13i - 2, $1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(u_{2i-1}u'_{2i-1})$ = k + 13i - 12, $1 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f^*(u_{2i}u'_{2i})$ = k + 13i - 8, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Clearly $f[V(G)] \cup \{f^*(e) : e \in E(G)\} = \{k, k+1, k+2, \ldots, p+q+k-1\}$.

An example of 2-super cube root cube mean labeling of $A(T_7) \odot K_1$[Triangle starts from u_2] is shown in Figure 3.

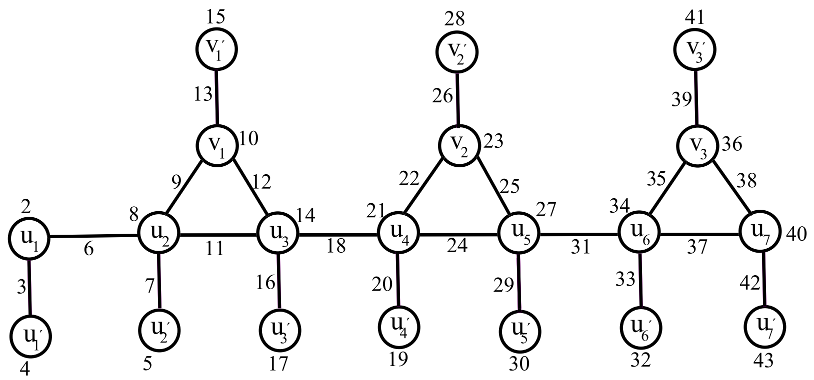

Figure 3: 2-super cube root cube mean labeling of A(T_7) ⊙ K_1 [Triangle starts from u_2]

From the above cases, A(T_n) ⊙ K_1 is a k-super cube root cube mean graph. □

Theorem 3.3. *The graph $A(T_n) \odot 2K_1$ is a k-super cube root cube mean graph.*

Proof. Let G = A(T_n) ⊙ 2K_1
Here consider two cases.
Case 1: The triangle starts from u_1.
Let V(G) = $\{u_i, u_i', u_i'' : 1 \leq i \leq n\} \cup \{v_i, v_i', v_i'' : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ and
E(G) = $\{u_iu_{i+1} : 1 \leq i \leq n\text{-}1\} \cup \{u_{2i-1}v_i, u_{2i}v_i, v_iv_i', v_iv_i'' : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$
$\cup \{ u_iu_i', u_iu_i'' : 1 \leq i \leq n \}$.

Here, p = $|V(G)| = \begin{cases} \frac{9n}{2}, & for\ n\ is\ even \\ \frac{9n-3}{2}, & for\ n\ is\ odd. \end{cases}$ and

q = $|E(G)| = \begin{cases} 5n-1, & for\ n\ is\ even \\ 5n-3, & for\ n\ is\ odd. \end{cases}$

Therefore p + q = $\begin{cases} \frac{19n-2}{2}, & for\ n\ is\ even \\ \frac{19n-9}{2}, & for\ n\ is\ odd. \end{cases}$

Now define a function $f : V(G) \rightarrow \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\ \}$ by
$f(u_{2i-1}) = k + 19i - 15,\ \ 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f(u_2) = \begin{cases} k+11, & k = 1,2,3,4,5,6; \\ k+12, & otherwise. \end{cases}$
$f(u_{2i}) = k + 19i - 7,\ \ 2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f(v_i) = k + 19i - 10,\ \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f(v'_i) = k + 19i - 14,\ \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f(v''_i) = k + 19i - 6,\ \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f(u'_{2i-1}) = k + 19i - 19,\ \ 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f(u''_{2i-1}) = k + 19i - 18,\ \ 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f(u'_{2i}) = k + 19i - 3,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f(u''_{2i}) = k + 19i - 2,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
Then the edge labels are
$f^*(u_{2i-1}u_{2i}) = k + 19i - 11,\ \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(u_{2i}u_{2i+1}) = k + 19i - 1,\ \ 1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(u_{2i-1}v_i) = k + 19i - 13,\ \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(u_{2i}v_i) = k + 19i - 9,\ \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(v_iv'_i) = k + 19i - 12,\ \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(v_1v''_1) = \begin{cases} k+12, & k = 1,2,3,4,5,6; \\ k+11, & otherwise. \end{cases}$
$f^*(v_iv''_i) = k + 19i - 8,\ \ 2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(u_{2i-1}u'_{2i-1}) = k + 19i - 17,\ \ 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f^*(u_{2i-1}u''_{2i-1}) = k + 19i - 16,\ \ 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f^*(u_{2i}u'_{2i}) = k + 19i - 5,\ \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(u_{2i}u''_{2i}) = k + 19i - 4,\ \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
Clearly $f[V(G)] \cup \{f^*(e) : e \in E(G)\} = \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$.
An example of 6-super cube root cube mean labeling of $A(T_6) \odot 2K_1$[Triangle start from u_2] is shown in Figure 4.

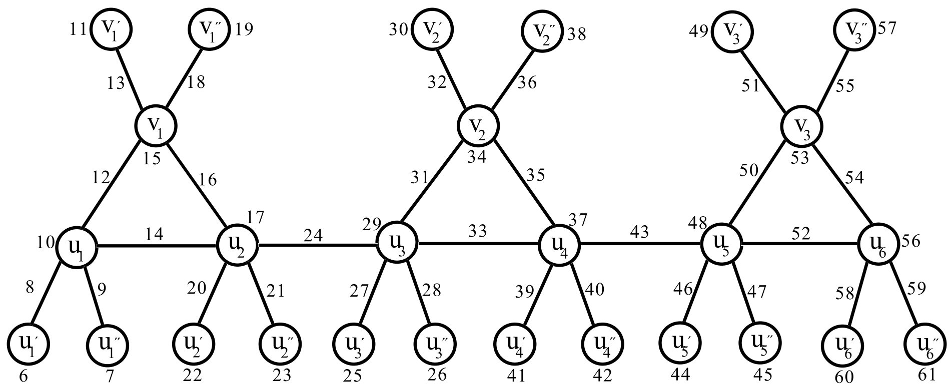

Figure 4: 6-super cube root cube mean labeling of $A(T_6) \odot 2K_1$[Triangle starts from u_1]

Case 2: The triangle starts from u_2.

Let V(G) = { u_i, u_i', u_i'' : $1 \le i \le n$ } ∪ { v_i, v_i', v_i'' : $1 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$ } and E(G) = { u_iu_{i+1} : $1 \le i \le n\text{-}1$} ∪ { $u_{2i}v_i$, $u_{2i+1}v_i$, v_iv_i', v_iv_i'' : $1 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$ } ∪ { u_iu_i', u_iu_i'' : $1 \le i \le n$ }.

Here p = $|V(G)| = \begin{cases} \frac{9n-6}{2}, & for\ n\ is\ even \\ \frac{9n-3}{2}, & for\ n\ is\ odd. \end{cases}$ and

q = $|E(G)| = \begin{cases} 5n-5, & for\ n\ is\ even \\ 5n-3, & for\ n\ is\ odd. \end{cases}$

Therefore p + q = $\begin{cases} \frac{19n-16}{2}, & for\ n\ is\ even \\ \frac{19n-9}{2}, & for\ n\ is\ odd. \end{cases}$

Now define a function f : V(G) → {k, k + 1, k + 2, . . . , p + q + k − 1 } by

$f(u_{2i-1})$ = k + 19i -17, $1 \le i \le \lceil \frac{n}{2} \rceil$.

$$f(u_2) = \begin{cases} k+8, & k = 1,2,3,4,5,6,7,8; \\ k+10, & otherwise. \end{cases}$$

$f(u_{2i})$ = k + 19i - 9, $2 \leq$ i $\leq \lfloor \frac{n}{2} \rfloor$.
$f(v_i)$ = k + 19i - 5, $1 \leq$ i $\leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f(v_i')$ = k + 19i -8, $1 \leq$ i $\leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f(v_i'')$ = k + 19i - 2, $1 \leq$ i $\leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f(u_{2i-1}')$ = k + 19i - 19, $1 \leq$ i $\leq \lceil \frac{n}{2} \rceil$.
$f(u_{2i-1}'')$ = k + 19i -15, $1 \leq$ i $\leq \lceil \frac{n}{2} \rceil$.
$f(u_{2i}')$ = k + 19i - 14, $1 \leq$ i $\leq \lfloor \frac{n}{2} \rfloor$.

$$f(u_2'') = \begin{cases} k+10, & k = 1,2,3,4,5,6,7,8; \\ k+8, & otherwise. \end{cases}$$

$f(u_{2i}'')$ = k + 19i - 11, $2 \leq$ i $\leq \lfloor \frac{n}{2} \rfloor$.
Then the edge labels are
$f^*(u_{2i-1}u_{2i})$ = k + 19i -13, $1 \leq$ i $\leq \lfloor \frac{n}{2} \rfloor$.
f^* $(u_{2i}u_{2i+1})$ = k + 19i - 3, $1 \leq$ i $\leq \left\lceil \frac{n-2}{2} \right\rceil$.
f^* $(u_{2i}v_i)$ = k + 19i - 7, $1 \leq$ i $\leq \left\lceil \frac{n-2}{2} \right\rceil$.
f^* $(u_{2i+1}v_i)$ = k + 19i - 1, $1 \leq$ i $\leq \left\lceil \frac{n-2}{2} \right\rceil$.
f^* (v_iv_i') = k + 19i - 6, $1 \leq$ i $\leq \left\lceil \frac{n-2}{2} \right\rceil$.
f^* (v_iv_i'') = k + 19i - 4, $1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.
f^* $(u_{2i-1}u_{2i-1}')$ = k + 19i - 18, $1 \leq$ i $\leq \lceil \frac{n}{2} \rceil$.
f^* $(u_{2i-1}u_{2i-1}'')$ = k + 19i -16, $1 \leq$ i $\leq \lceil \frac{n}{2} \rceil$.
f^* $(u_{2i}u_{2i}')$ = k + 19i - 12, $1 \leq$ i $\leq \lfloor \frac{n}{2} \rfloor$.
f^* $(u_{2i}u_{2i}'')$ = k + 19i - 10, $1 \leq$ i $\leq \lfloor \frac{n}{2} \rfloor$.

Clearly $f[V(G)] \cup \{f^*(e) : e \in E(G)\} = \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$.

An example of 6-super cube root cube mean labeling of $A(T_5) \odot 2K_1$[Triangle starts from u_2] is shown in Figure 5.

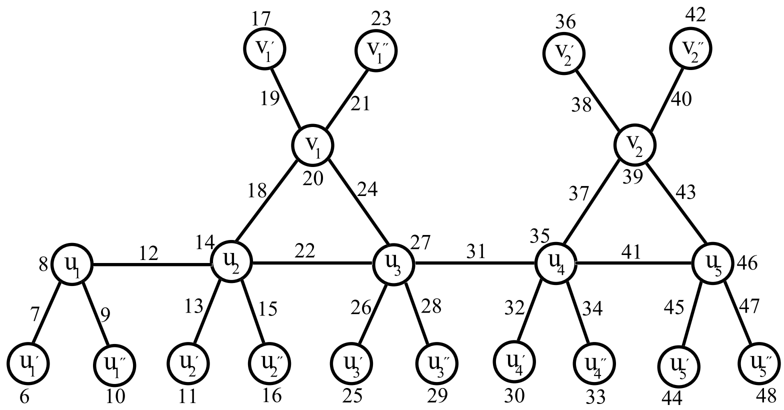

Figure 5: 6-super cube root cube mean labeling of A(T_5) ⊙ 2K_1[Triangle starts from u_2]

From the above cases, A(T_n) ⊙ 2K_1 is a k-super cube root cube mean graph. □

Theorem 3.4. *The graph A(Q_n) ⊙ K_1 is a k-super cube root cube mean graph.*

Proof. Let G = A(Q_n) ⊙ K_1
Here consider two cases.

Case 1: Quadrilateral starts from u_1.

Let V(G) = { u_i, u'_i : $1 \leq i \leq n$ } ∪ { v_i, w_i, v'_i, w'_i: $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ } and E(G) = { u_iu_{i+1} : $1 \leq i \leq n-1$} ∪ { $u_{2i-1}v_i$, $u_{2i}w_i$, $v_iv'_i$, v_iw_i, $w_iw'_i$: $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ } ∪ { $u_iu'_i$: $1 \leq i \leq n$ }.

here p = $\begin{cases} 4n, & for\ n\ is\ even \\ 4n-2, & for\ n\ is\ odd. \end{cases}$ & q = $\begin{cases} \frac{9n-2}{2}, & for\ n\ is\ even \\ \frac{9n-7}{2}, & for\ n\ is\ odd. \end{cases}$

Therefore p + q = $\begin{cases} \frac{17n-2}{2}, & for\ n\ is\ even \\ \frac{17n-11}{2}, & for\ n\ is\ odd. \end{cases}$

Now define a function
$f : V(G) \rightarrow \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\ \}$ by
$f(u_{2i-1}) = k + 17i - 15,\ 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f(u_2) = \begin{cases} k+12, & k = 1 \\ k+13, & otherwise. \end{cases}$

$f(u_{2i}) = k + 17i - 4,\ 2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f(w_1) = \begin{cases} k+8, & k = 1,2,3,\ldots,11; \\ k+9, & otherwise. \end{cases}$

$f(w_i) = k + 17i - 8,\ 2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f(v_i) = k + 17i - 11,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f(v'_i) = k + 17i - 14,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f(w'_1) = \begin{cases} k+13, & k = 1; \\ k+12, & otherwise. \end{cases}$
$f(w'_i) = k + 17i - 5,\ 2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f(u'_{2i-1}) = k + 17i - 17,\ 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f(u'_{2i}) = k + 17i - 2,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
Then the edge labels are

$f^*(u_1u_2) = \begin{cases} k+9, & k = 1,\ 2,\ 3, \ldots, 11 \\ k+8, & otherwise. \end{cases}$

$f^*(u_{2i-1}u_{2i}) = k + 17i - 9,\ 2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(u_{2i}u_{2i+1}) = k + 17i - 1,\ 1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(u_{2i-1}v_i) = k + 17i - 13,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(u_{2i}w_i) = k + 17i - 6,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(v_iw_i) = k + 17i - 10,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(v_iv'_i) = k + 17i - 12,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(w_iw'_i) = k + 17i - 7,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
$f^*(u_{2i-1}u_{2i}{}'_{-1}) = k + 17i - 16,\ 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
$f^*(u_{2i}u'_{2i}) = k + 17i - 3,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
Clearly $f[V(G)] \cup \{f^*(e) : e \in E(G)\} = \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$.

An example of 2-super cube root cube mean labeling of $A(Q_6) \odot K_1$[Quadrilateral starts from u_1] is shown in Figure 6.

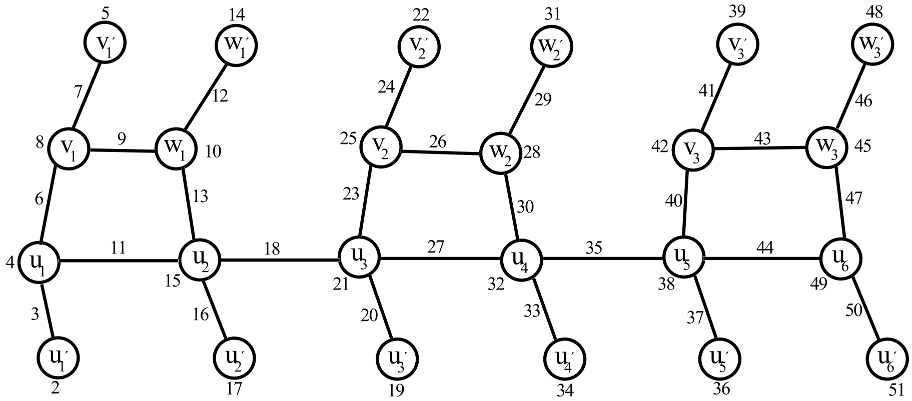

Figure 6: 2-super cube root cube mean labeling of $A(Q_6) \odot K_1$[Quadrilateral starts from u_1]

Case 2: Quadrilateral starts from u_2.

Let $V(G) = \{ u_i,\ u'_i : 1 \leq i \leq n \} \cup \{ v_i,\ w_i,\ v'_i,\ w'_i : 1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil \}$ and $E(G) = \{ u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{ u_{2i}v_i,\ u_{2i+1}w_i,\ v_iv'_i,\ v_iw_i,\ w_iw'_i : 1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil \} \cup \{ u_iu'_i : 1 \leq i \leq n \}$.

here $p = \begin{cases} 4n-4, & for\ n\ is\ even \\ 4n-2, & for\ n\ is\ odd. \end{cases}$ & $q = \begin{cases} \frac{9n-12}{2}, & for\ n\ is\ even \\ \frac{9n-7}{2}, & for\ n\ is\ is\ odd. \end{cases}$

Therefore $p + q = \begin{cases} \frac{17n-20}{2}, & for\ n\ is\ even \\ \frac{17n-11}{2}, & for\ n\ is\ odd. \end{cases}$

Now define a function

$f : V(G) \rightarrow \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$ by

$f(u_{2i-1}) = k + 17i - 17,\ 1 \leq i \leq \lceil \frac{n}{2} \rceil$.

$f(u_{2i}) = k + 17i - 11,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

$f(v_i) = k + 17i - 7,\ 1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil$.

$f(w_1) = \begin{cases} k+12, & k = 1,\ 2, \ldots,\ 7 \\ k+13, & otherwise. \end{cases}$

$f(w_i) = k + 17i - 4,\ \ 2 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$.
$f(v'_i) = k + 17i - 10,\ \ 1 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$.
$f(w'_i) = k + 17i - 1,\ \ 1 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$.
$f(u'_{2i-1}) = k + 17i - 15,\ \ 1 \le i \le \lceil \frac{n}{2} \rceil$.
$f(u'_2) = \begin{cases} k+3, \ k=1,\ 2,3,4; \\ k+4, \ otherwise. \end{cases}$
$f(u'_{2i}) = k + 17i - 13,\ \ 2 \le i \le \lfloor \frac{n}{2} \rfloor$.
Then the edge labels are
$f^*(u_1u_2) = \begin{cases} k+4, \ k=1,\ 2,3,4; \\ k+3, \ otherwise. \end{cases}$

$f^*(u_{2i-1}u_{2i}) = k + 17i - 14,\ \ 2 \le i \le \lfloor \frac{n}{2} \rfloor$.
$f^*(u_2u_3) = \begin{cases} k+13, \ k=1,\ 2,3,\ldots,\ 7; \\ k+12, \ otherwise. \end{cases}$
$f^*(u_{2i}u_{2i+1}) = k + 17i - 5,\ \ 2 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(u_{2i}v_i) = k + 17i - 9,\ \ 1 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(u_{2i+1}w_i) = k + 17i - 2,\ \ 1 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(v_iw_i) = k + 17i - 6,\ \ 1 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(v_iv'_i) = k + 17i - 8,\ \ 1 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(w_iw'_i) = k + 17i - 3,\ \ 1 \le i \le \left\lceil \frac{n-2}{2} \right\rceil$.
$f^*(u_{2i-1}u'_{2i-1}) = k + 17i - 16,\ \ 1 \le i \le \lceil \frac{n}{2} \rceil$.
$f^*(u_{2i}u'_{2i}) = k + 17i - 12,\ \ 1 \le i \le \lfloor \frac{n}{2} \rfloor$.
Clearly $f[V(G)] \cup \{f^*(e) : e \in E(G)\} = \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$.

An example of 2-super cube root cube mean labeling of $A(Q_5) \odot K_1$[Quadrilateral starts from u_2] is shown in Figure 7.

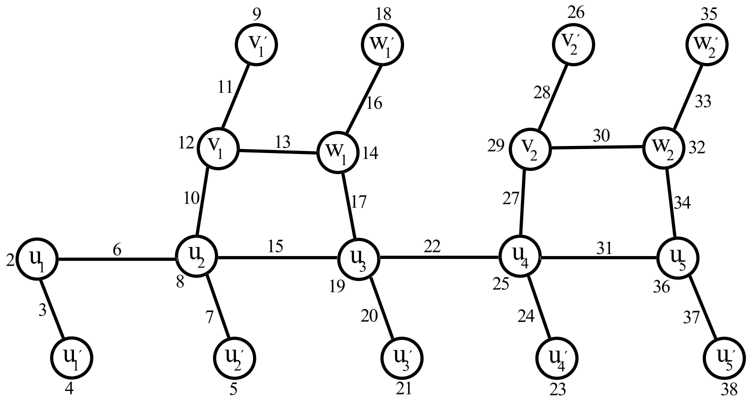

Figure 7: 2-super cube root cube mean labeling of A(Q_5) $\odot$ K_1[Quadrilateral starts from u_2]

From the above cases, A(Q_n) $\odot$ K_1 is a k-super cube root cube mean graph. □

Theorem 3.5. *The graph $P_n \odot K_{1,2}$ is a k-super cube root cube mean graph.*

Proof. Let G = $P_n \odot K_{1,2}$
Let V(G) = $\{u_i, v_i, w_i, 1 \leq i \leq n\}$ and
E(G) = $\{u_i v_i, u_i w_i, 1 \leq i \leq n\} \cup \{u_i u_{i+1}, 1 \leq i \leq n\text{-}1\}$
Here $p = |V(G)| = 3n$ and $q = |E(G)| = 3n - 1$
Hence $p + q = 6n - 1$.
Now define a function
$f : V(G) \rightarrow \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$ by
$f(u_i) = k + 6i - 4, 1 \leq i \leq n$
$f(v_i) = k + 6i - 6, 1 \leq i \leq n$ and $i \neq 2$
$$f(v_2) = \begin{cases} k+5, & k = 1, 2; \\ k+6, & \text{otherwise.} \end{cases}$$
$f(w_i) = k + 6i - 2, 1 \leq i \leq n$

Then the edge labels are
$f^*(u_i v_i) = k + 6i - 5,\ 1 \le i \le n$
$f^*(u_i w_i) = k + 6i - 3,\ 1 \le i \le n$
$f^*(u_1\ u_2) = \begin{cases} k+6, & k = 1,2; \\ k+5, & otherwise. \end{cases}$
$f^*(u_i u_{i+1}) = k + 6i - 1, 2 \le i \le n-1$
Hence $f[V(G)] \cup \{f^*(e) : e \in E(G)\} = \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\}$.
Therefore $P_n \odot K_{1,2}$ is a k-super cube root cube mean graph.
An example of 12-super cube root cube mean labeling of $P_3 \odot K_{1,2}$ is shown in Figure 8.

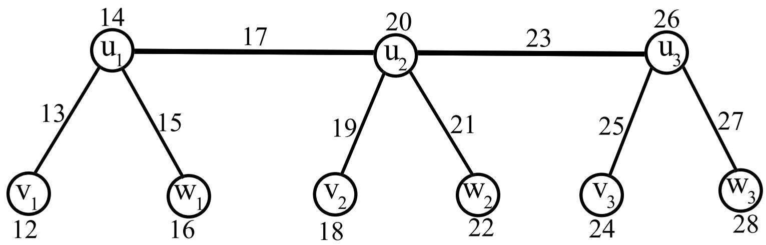

Figure 8: 12-super cube root cube mean labeling of $P_3 \odot K_{1,2}$

□

Theorem 3.6. *The graph $P_n \odot K_{1,3}$ is a k-super cube root cube mean graph.*

Proof. Let $G = P_n \odot K_{1,3}$
Let $V(G) = \{u_i,\ v_i,\ w_i,\ s_i,\ 1 \le i \le n\}$ and
$E(G) = \{u_i v_i,\ u_i w_i,\ u_i s_i,\ 1 \le i \le n\} \cup \{u_i u_{i+1},\ 1 \le i \le n-1\}$
Here $p = |V(G)| = 4n$ and $q = |E(G)| = 4n - 1$
Hence $p + q = 8n - 1$.
Now define a function
$f : V(G) \to \{k,\ k+1,\ k+2, \ldots,\ p+q+k-1\ \}$ by
$f(u_i) = k + 8i - 4, 1 \le i \le n$
$f(v_1) = k$, for all k
$f(v_i) = k + 8i - 9, 2 \le i \le n$
$f(w_i) = k + 8i - 7, 1 \le i \le n$ and $i \ne 2$
$f(w_2) = \begin{cases} k+8, & k = 1,2,3,4,5,6; \\ k+9, & otherwise. \end{cases}$

$f(s_i) = k + 8i - 2, 1 \leq i \leq n.$
Then the edge labels are

$$f^*(u_1u_2) = \begin{cases} k+9, & k = 1,2,3,4,5,6; \\ k+8, & otherwise. \end{cases}$$

$f^*(u_iu_{i+1}) = k + 8i, 2 \leq i \leq n - 1$
$f^*(u_iv_i) = k + 8i-6, 1 \leq i \leq n$
$f^*(u_iw_i) = k + 8i-5, 1 \leq i \leq n$
$f^*(u_is_i) = k + 8i-3, 1 \leq i \leq n.$
Hence $f[V(G)] \cup \{f^*(e) : e \in E(G)\} = \{k, k+1, k+2, \ldots, p+q+k-1\}$.
Therefore $P_n \odot K_{1,3}$ is a k-super cube root cube mean graph.
An example of 5-super cube root cube mean labeling of $P_3 \odot K_{1,3}$ is shown in Figure 9.

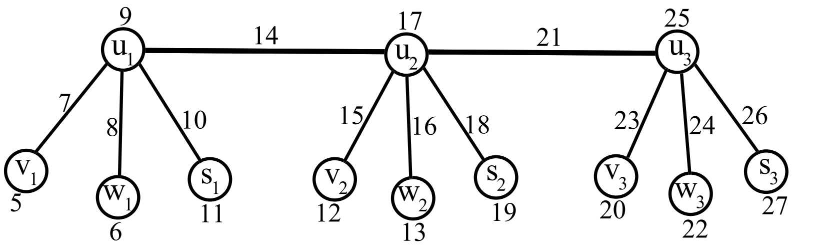

Figure 9: 5-super cube root cube mean labeling of $P_3 \odot K_{1,3}$

□

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